

STRONG CONVERGENCE OF MODIFIED MANN ITERATIONS FOR LIPSCHITZ PSEUDOCONTRACTIONS

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Abstract

In this paper, for a Lipschitz pseudocontractive mapping T , we study the strong convergence of the iterative scheme generated by

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n((1 - \beta_n)u + \beta_nTx_n),$$

when $\{\beta_n\}, \{\alpha_n\}$ satisfy (i) $\lim_{n \rightarrow \infty} \alpha_n = 0$; (ii) $\sum_{n=1}^{\infty} \alpha_n = \infty$; (iii) $\lim_{n \rightarrow \infty} \beta_n = 0$.

2000 Mathematics Subject Classification: 49J30, 47H10, 47H17, 90C99, 47H06, 47J05, 47J25, 47H09.

Keywords and phrases: Lipschitz pseudocontractions, modified Mann iterations, strong convergence, uniformly Gâteaux differentiable norm.

Received October 22, 2008

1. Introduction

Let K be a nonempty closed convex subset of a smooth Banach space E . Let $T : K \rightarrow K$ be a continuous pseudocontractive mapping. Then the mapping $T_t = tu + (1 - t)T$ obviously is a continuous strongly pseudocontractive mapping from K to K for each fixed $t \in (0, 1)$. Therefore, T_t has a unique fixed point in K ([14, Corollary 2]), i.e., for any given $t \in (0, 1)$, there exists $x_t \in K$ such that

$$x_t = tu + (1 - t)Tx_t. \quad (1.1)$$

As $t \rightarrow 0$, the strong convergence of the path $\{x_t\}$ has been introduced and studied by Browder [1] for a nonexpansive mapping T in Hilbert space, by Reich [23] for a nonexpansive self-mapping T defined on a uniformly smooth Banach space, by Takahashi-Ueda [36] for a nonexpansive self-mapping T defined on a uniformly convex Banach space with a uniformly *Gâteaux* differentiable norm, by Xu [39] for a nonexpansive self-mapping T defined on a reflexive Banach space which has a weakly continuous duality mapping J_ϕ , by Schu [27] for a continuous pseudocontractive nonself-mapping defined on a reflexive Banach space having weakly sequentially continuous duality mapping, by Morales-Jung [22] and Udomene [37] for a continuous pseudocontractive mapping T satisfying the weakly inward condition and defined on a (reflexive) Banach space with a uniformly *Gâteaux* differentiable norm. See also Bruck [2, 3], Reich [23, 25], Song et al. [5, 28, 32, 33], Suzuki [35], and others.

On the other hand, Mann [21] introduced the following iteration for T in a Hilbert space:

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)Tx_n, \quad n \geq 0, \quad (1.2)$$

where $\{\alpha_n\}$ is a sequence in $[0, 1]$. Latterly, Reich [24] studied this iteration in a uniformly convex Banach space with a Fréchet differentiable norm and obtained its weak convergence. In the last twenty years of so, numerous papers have been published on the iterative approximation of fixed points of Lipschitz strongly pseudocontractive (and correspondingly Lipschitz strongly accretive) mappings using the

Mann iteration process. Results which had been known only for Hilbert spaces and only for Lipschitz mappings have been extended to more general Banach spaces (see, e.g., [4, 6, 8-13, 15-17, 20, 31] and others). This success, however, has not carried over to arbitrary Lipschitz pseudocontraction T even when the domain of the operator T is compact convex subset of a Hilbert space. In fact, it was a long standing open question whether or not the Mann iteration process converges under this setting. In 1999, Chidume-Moore [9] proposed the following problem in connection with the iterative approximation of fixed points of pseudocontractions.

Open Problem. Does the Mann iteration process always converge for continuous pseudo-contractions, or for even Lipschitz pseudo-contractions?

These questions have recently been resolved in the negative by Chidume-Mutangadura [8], who produced an example of a Lipschitz pseudo-contractive map defined on a compact convex subset of real Hilbert space with a unique fixed point for which no Mann sequence converges. In order to get a strong convergence result, one has to modify the normal Mann's iteration algorithm. Some attempts have been made and several important results have been reported. For example, Kim-Xu [18], Chidume-Chidume [7], Suzuki [34] and Song-Chen [29] dealt with strong convergence of the modified Mann iteration (1.3) for a nonexpansive mapping T : for $x_0, u \in \mathbf{K}$,

$$x_{n+1} = \alpha_n u + (1 - \alpha_n)(\beta_n T x_n + (1 - \beta_n)x_n). \quad (1.3)$$

Recently, Chidume-Ofoedu [6] and Song [30] tried to found the strong convergence of the following iteration $\{x_n\}$ which independent of the path x_t ;

$$x_{n+1} = (1 - \lambda_n - \lambda_n \theta_n)x_n + \lambda_n \theta_n x_1 + \lambda_n (\alpha_n T x_n + (1 - \alpha_n)x_n), \quad n \geq 1. \quad (1.4)$$

Very recently, Zhou [40] obtained the strong convergence theorem of the iterative sequence (1.5) for λ -strict pseudocontraction T in 2-uniformly smooth Banach space: for $u, x_0 \in E$,

$$x_{n+1} = \beta_n u + \gamma_n x_n + (1 - \beta_n - \gamma_n)(\alpha_n x_n + (1 - \alpha_n)Tx_n), \quad n \geq 0. \quad (1.5)$$

It is the purpose of this paper to present the modified Mann's iteration algorithm for a Lipschitz pseudocontraction T . Namely, a new algorithm will be proposed to find a fixed point of T . Our method is different from the above several iterations where the mapping T involving with the algorithm is averaged. Instead, our algorithm proposed below works for every Lipschitz pseudocontraction T . More precisely, I will replace the point Tx_n in algorithm (1.2) with $(1 - \beta_n)u + \beta_n Tx_n$ at step n , for $\beta_n \in (0, 1)$. That is, my method produces a sequence $\{x_n\}$ according to the iteration process:

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n((1 - \beta_n)u + \beta_n Tx_n), \quad (1.6)$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences in $(0, 1)$ satisfying the

conditions: (i) $\lim_{n \rightarrow \infty} \alpha_n = 0$; (ii) $\sum_{n=1}^{\infty} \alpha_n = \infty$; (iii) $\lim_{n \rightarrow \infty} \beta_n = 0$. It will be

proved that $\{x_n\}$ strongly converges to some fixed point of a Lipschitz pseudocontraction T . In particular, the parameters of our iterative sequence are simpler and don't depend on each other.

2. Preliminaries

Throughout this paper, a Banach space E will always be over the real scalar field. We denote its norm by $\|\cdot\|$ and its dual space by E^* . The value of $x^* \in E^*$ at $y \in E$ is denoted by $\langle y, x^* \rangle$, and the normalized duality mapping from E into 2^{E^*} is denoted by J , that is, $J(x) = \{f \in E^* : \langle x, f \rangle = \|x\|\|f\|, \|x\| = \|f\|\}$, $\forall x \in E$. Let $F(T) = \{x \in E : Tx = x\}$, the set of all fixed point for a mapping T .

Let $S(E) := \{x \in E; \|x\| = 1\}$ denote the unit sphere of a Banach space E . The space E is said to have (i) *a uniformly Gâteaux differentiable norm*, if for each y in $S(E)$, the limit $\lim_{t \rightarrow 0} \frac{\|x+ty\| - \|x\|}{t}$ is uniformly attained for $x \in S(E)$; (ii) *fixed point property* for non-expansive self-mappings, if

each non-expansive self-mapping defined on any bounded closed convex subset K of E has at least a fixed point. A Banach space E is said to be *strictly convex* if $\|x\| = \|y\| = 1$, $x \neq y$ implies $\|\frac{x+y}{2}\| < 1$.

Recall that a mapping T with domain $D(T)$ and range $R(T)$ in Banach space E is called *strongly pseudo-contractive* if, for all $x, y \in D(T)$, there exist $k \in (0, 1)$ and $j(x-y) \in J(x-y)$ such that

$$\langle Tx - Ty, j(x-y) \rangle \leq k\|x-y\|^2. \quad (2.1)$$

While T is said to be *pseudo-contractive* if (2.1) holds for $k = 1$. T is said to be *Lipschitzian* if, for all $x, y \in D(T)$, there exists $L > 0$ such that

$$\|Tx - Ty\| \leq L\|x - y\|.$$

The mapping T is called *non-expansive* if $L = 1$ and, further, T is said to be *contractive* if $L < 1$. It is obvious that (contractive) nonexpansive mapping is an important subclass of (Lipschitz strongly) pseudocontractive mapping, but the converse implication may be false. This can be seen from the existing examples (see, e.g., [6, 8, 40]).

If C and D are nonempty subsets of a Banach space E such that C is nonempty closed convex and $D \subset C$, then a mapping $P : C \rightarrow D$ is called a *retraction* from C to D if P is continuous with $F(P) = D$. A mapping $P : C \rightarrow D$ is called *sunny* if $P(Px + t(x - Px)) = Px$, $\forall x \in C$ whenever $Px + t(x - Px) \in C$ and $t > 0$. A subset D of C is said to be a *sunny nonexpansive retract* of C if there exists a sunny nonexpansive retraction of C onto D . The term ‘‘sunny nonexpansive retraction’’ was coined by Reich in [26]. For more details, see [19, 25, 26].

Lemma 2.1 ([22, 32, 33, 36]). *Let E be a reflexive Banach space which has both fixed point property for non-expansive self-mappings and a uniformly Gâteaux differentiable norm or be a reflexive and strictly convex Banach space with a uniformly Gâteaux differentiable norm, and K be a nonempty closed convex subset of E . Suppose that T is a continuous*

pseudocontractive mapping from K into K with $F(T) \neq \emptyset$. Then as $t \rightarrow 0$, x_t , defined by

$$x_t = tu + (1-t)Tx_t$$

converges strongly to a fixed point P_u of T , where P is the unique sunny nonexpansive retract from K onto $F(T)$.

Lemma 2.2 (Liu [20] and Xu [38]). Let $\{a_n\}$ be a sequence of nonnegative real numbers satisfying the property:

$$a_{n+1} \leq (1-t_n)a_n + b_n + t_nc_n, \quad \forall n \geq 0,$$

where $\{t_n\}$, $\{b_n\}$, $\{c_n\}$ satisfy the restrictions:

$$(i) \sum_{n=0}^{\infty} t_n = \infty; (ii) \sum_{n=0}^{\infty} b_n < +\infty; (iii) \limsup_{n \rightarrow \infty} c_n \leq 0.$$

Then $\{a_n\}$ converges to zero as $n \rightarrow \infty$.

3. Main Results

Theorem 3.1. Let E be a reflexive Banach space which has both fixed point property for non-expansive self-mappings and a uniformly Gâteaux differentiable norm, and K be a nonempty closed convex subset of E . Suppose $T : K \rightarrow K$ is a Lipschitzian pseudo-contraction with a Lipschitz constant $L > 0$ and $F(T) \neq \emptyset$, and $\{x_n\}$ is a sequence given by

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n((1-\beta_n)u + \beta_nTx_n). \quad (3.1)$$

Assume that $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences in $(0, 1)$ satisfying the conditions:

$$(ii) \lim_{n \rightarrow \infty} \alpha_n = 0; (ii) \sum_{n=1}^{\infty} \alpha_n = \infty; (iii) \lim_{n \rightarrow \infty} \beta_n = 0.. Then, as $n \rightarrow \infty$, $\{x_n\}$$$

converges strongly to some fixed point P_u of T , where P is the unique sunny nonexpansive retract from K onto $F(T)$.

Proof. The proof will be split into three steps.

Step 1. $\{x_n\}$ is bounded. Take $p \in F(T)$. Choose $M > 0$ sufficiently large such that

$$\|x_1 - p\| \leq M, \quad \|u - p\| \leq \frac{M}{L}.$$

We proceed by induction to show that $\|x_n - p\| \leq M$ for all $n \geq 1$. Assume that $\|x_n - p\| \leq M$ for some $n > 1$. We show that $\|x_{n+1} - p\| \leq M$. In fact, from (3.1), we estimate as follows:

$$\begin{aligned} \|x_{n+1} - p\| &\leq \alpha_n(1 - \beta_n)\|u - p\| + \alpha_n\beta_n\|Tx_n - p\| + (1 - \alpha_n)\|x_n - p\| \\ &\leq \alpha_n(1 - \beta_n)\frac{M}{L} + \alpha_n\beta_nLM + (1 - \alpha_n)M. \end{aligned}$$

Then when $L = 1$, the result is obvious. Below let $L > 1$. We use the reduction to absurdity. Suppose that $\|x_{n+1} - p\| > M$. We have

$$M < \alpha_n(1 - \beta_n)\frac{M}{L} + \alpha_n\beta_nLM + (1 - \alpha_n)M.$$

Then $0 < \alpha_n(1 - \beta_n)\frac{1}{L} + \alpha_n\beta_nL - \alpha_n$, and hence

$$0 < 1 - \beta_n + \beta_nL^2 - L, \text{ that is, } \frac{1}{L+1} = \frac{L-1}{L^2-1} < \beta_n.$$

This contradicts to the conditions $\lim_{n \rightarrow \infty} \beta_n = 0$. Therefore, $\|x_{n+1} - p\| \leq M$ for all n . This proves the boundedness of the sequence $\{x_n\}$, which implies that the sequence $\{Tx_n\}$ is also bounded.

Step 2.

$$\limsup_{n \rightarrow \infty} \langle u - Pu, J(x_{n+1} - Pu) \rangle \leq 0. \quad (3.2)$$

Let T_l be defined by $T_lx := (1 - \alpha_l)x + \alpha_lTx$ for each $x \in K$ and fixed $\alpha_l \in (0, 1)$. Then, we observe that for each l , T_l is a Lipschitz pseudocontractive mapping from K to itself with Lipschitz constant

$1 + \alpha_n(L - 1) > 0$ and $F(T_l) = F(T)$. Moreover, the definition of T_l reduces to

$$\lim_{l \rightarrow \infty} \|T_l x_n - x_n\| = \lim_{l \rightarrow \infty} \alpha_l \|Tx_n - x_n\| = 0. \quad (3.3)$$

Setting $T_t^l = tu + (1 - t)T_l$, then for each $t \in (0, 1)$ and $\alpha_l \in (0, 1)$, T_t^l obviously is a continuous strongly pseudocontractive mapping from K to K for each $t \in (0, 1)$ and each l . Therefore, T_t^l has a unique fixed point in K (see [14, Corollary 2]), that is, for each $t \in (0, 1)$ and each l ,

$$z_t^l = tu + (1 - t)T_l z_t^l.$$

Then for each l , it follows from Lemma 2.1 that $\lim_{t \rightarrow 0} z_t^l = P_l u$, is the unique sunny nonexpansive retraction from K onto $F(T_l) = F(T)$. Then $P_l \equiv P$ by the uniqueness of sunny nonexpansive retraction from K onto $F(T)$, and hence

$$\lim_{t \rightarrow 0} z_t^l = Pu \text{ for all } l. \quad (3.4)$$

Since T_l is a pseudocontractive mapping for each l , using the equality

$$z_t^l - x_l = (1 - t)(T_l z_t^l - x_n) + t(u - x_n),$$

we have

$$\begin{aligned} \|z_t^l - x_n\|^2 &= (1 - t)\langle T_l z_t^l - x_n, J(z_t^l - x_n) \rangle + t\langle u - x_n, J(z_t^l - x_n) \rangle \\ &= (1 - t)(\langle T_l z_t^l - T_l x_n, J(z_t^l - x_n) \rangle + \langle T_l x_n - x_n, J(z_t^l - x_n) \rangle) \\ &\quad + t\langle u - z_t^l, J(z_t^l - x_n) \rangle + t\|z_t^l - x_n\|^2 \\ &\leq \|x_n - z_t^l\|^2 + \|T_l x_n - x_n\| \|J(z_t^l - x_n)\| + t\langle u - z_t^l, J(z_t^l - x_n) \rangle, \end{aligned}$$

and hence,

$$\langle u - z_t^l, J(x_n - z_t^l) \rangle \leq \frac{\|T_l x_n - x_n\|}{t} C,$$

for some constant $C > 0$. Hence, noting (3.3), we obtain

$$\limsup_{l \rightarrow \infty} \langle u - z_t^l, J(x_n - z_t^l) \rangle \leq 0.$$

Therefore, for any $\varepsilon > 0$, there exists a positive integer N such that for all $l \geq N$,

$$\langle u - z_t^l, J(x_n - z_t^l) \rangle < \frac{\varepsilon}{2}. \tag{3.5}$$

On the other hand, since J is norm topology to weak* topology uniformly continuous on bounded sets and $\lim_{t \rightarrow 0} \|z_t^N - Pu\| = 0$, we have

$$\begin{aligned} & |\langle u - Pu, J(x_n - Pu) \rangle - \langle u - z_t^N, J(x_n - z_t^N) \rangle| \\ &= |\langle u - Pu, J(x_n - Pu) - J(x_n - z_t^N) \rangle + \langle z_t^N - Pu, J(x_n - z_t^N) \rangle| \\ &\leq |\langle u - Pu, J(x_n - Pu) - J(x_n - z_t^N) \rangle| + \|z_t^N - Pu\|M \rightarrow 0, \text{ as } t \rightarrow 0. \end{aligned}$$

Hence, for the above $\varepsilon > 0$, $\exists \delta > 0$, such that $\forall t \in (0, \delta)$, for all n , we have

$$\langle u - Pu, J(x_n - Pu) \rangle \leq \langle u - z_t^N, J(x_n - z_t^N) \rangle + \frac{\varepsilon}{2}.$$

By (3.5), we have that

$$\limsup_{n \rightarrow \infty} \langle u - Pu, J(x_n - Pu) \rangle < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Since ε is arbitrary, (3.2) is proved.

Step 3. $\lim_{n \rightarrow \infty} \|x_n - Pu\| = 0$.

From (3.1), we have

$$\begin{aligned} & \|x_{n+1} - Pu\|^2 \\ &= \langle \alpha_n(1 - \beta_n)(u - Pu) + \beta_n \alpha_n(Tx_n - Pu) + (1 - \alpha_n)(x_n - Pu), J(x_{n+1} - Pu) \rangle \\ &\leq \alpha_n(1 - \beta_n) \langle u - Pu, J(x_{n+1} - Pu) \rangle + \beta_n \alpha_n \|Tx_n - Pu\| \|x_{n+1} - Pu\| \end{aligned}$$

$$\begin{aligned}
& +(1 - \alpha_n)\langle x_n - Pu, J(x_{n+1} - Pu) \rangle \\
\leq & (1 - \alpha_n) \frac{\|x_n - Pu\|^2}{2} + \frac{\|x_{n+1} - Pu\|^2}{2} + \beta_n \alpha_n LM^2 \\
& + \alpha_n(1 - \beta_n)\langle u - Pu, J(x_{n+1} - Pu) \rangle,
\end{aligned}$$

which implies that

$$\|x_{n+1} - Pu\|^2 \leq (1 - \alpha_n)\|x_n - Pu\|^2 + \alpha_n \theta_n, \quad (3.6)$$

where $\theta_n = 2\beta_n LM^2 + 2\alpha_n(1 - \beta_n)\langle u - Pu, J(x_{n+1} - Pu) \rangle$.

Using the condition (iii) and (3.2), we have $\limsup_{n \rightarrow \infty} \theta_n \leq 0$. Hence, Applying Lemma 2.2 to the inequality (3.6), we conclude that $\lim_{n \rightarrow \infty} \|x_n - Pu\| = 0$. This completes the proof.

We remark that if E is strictly convex, then the property that E has the fixed point property for nonexpansive self-mappings may be dropped. In fact, we have the following theorem.

Theorem 3.2. *Let E be a reflexive and strictly convex Banach space with a uniformly Gâteaux differentiable norm. suppose $K, T, \{x_n\}, \{\alpha_n\}, \{\beta_n\}$ are as Theorem 3.1. Then as $n \rightarrow \infty, \{x_n\}$ converges strongly to some fixed point Pu of T , where P is the unique sunny nonexpansive retract from K onto $F(T)$.*

Proof. This follows from Lemma 2.1 and the proof of Theorem 3.1

As a direct consequence of Theorems 3.1 and 3.2, we obtain the following corollaries.

Corollary 3.3. *Let E be a reflexive Banach space which has both fixed point property for non-expansive self-mappings and a uniformly Gâteaux differentiable norm or be a reflexive and strictly convex Banach space with a uniformly Gâteaux differentiable norm. Suppose $K, f, \{x_n\}, \{\alpha_n\}, \{\beta_n\}$ are as Theorem 3.1, and $T : K \rightarrow K$ is a nonexpansive mapping. Then, as $n \rightarrow \infty, \{x_n\}$ converges strongly to some fixed point Pu of T , where P is the unique sunny nonexpansive retract from K onto $F(T)$.*

Remark 1. (i) There many Banach spaces which has fixed point property for non-expansive self-mappings. For example, compact Banach space, uniformly convex Banach space, uniformly smooth Banach space, reflexive Banach space with the Opial's property, reflexive Banach space with normal structure and so on.

(ii) It is easy to find examples of spaces which satisfy the fixed point property for non-expansive self-mappings, which are not strictly convex. On the other hand, it appears to be unknown whether a reflexive and strictly convex Banach space satisfies the fixed point property for nonexpansive self-mappings (see [22]).

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